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Aspects of achievable performance for quarter-car active suspensions ☆

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Abstract

In this paper, constraints on the transfer functions from the road disturbance to the vertical acceleration, the suspension travel, and the tire deflection are derived for a quarter-car active suspension system using the vertical acceleration and/or the suspension travel measurements for feedback. The derived constraints complement the similar constraints in the literature. By using the factorization approach to feedback stability, it is shown that tire damping couples the motions of the sprung and unsprung masses; and eliminates a constraint at the wheel-hop frequency. The influence of tire damping on the design of an active suspension system for a quarter-car model by a mixture of the LQG methodology and the interpolation approach is also illustrated.

1. Introduction

The suspension system is the main tool to achieve ride comfort and drive safety for a vehicle. Passive suspension systems have been designed to obtain a good compromise between these objectives, but intrinsic limitations prevent them from obtaining the best performances for both goals. Compared with passive suspension systems, active and semi-active suspension systems can achieve a better compromise during various driving conditions.

Active and semi-active control of vehicle suspensions have been the subject of considerable investigation since the late 1960s; see, for example Refs. [1–12] and the references therein. Constraints and trade-offs on achievable performances have been studied in Refs. [13–17]. As put forward in Ref. [16], in a study of constraints and trade-offs from a control systems point of view, one has to properly address: (i) What can and cannot be achieved with general dynamic compensation, and (ii) How much freedom is gained by the selection of measurements for feedback purpose?

In Refs. [13,14], constraints on achievable frequency responses were derived from an *invariant point* perspective. A framework using mechanical multi-port networks to study the performance capabilities and constraints is developed in Ref. [17]. In Ref. [16], for a quarter-car model of an automative suspension a

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complete set of constraints on several transfer functions of interest from the road and the load disturbances were determined by making use of the *factorization* approach to feedback stability and the Youla *parameterization* of stabilizing controllers. Roughly speaking, completeness means that from a given set of constraints, one can identify a quarter-car model within the model class matching the given constraints. Such an approach reveals the degrees of freedom in shaping the response of the vehicle to disturbances and determines a minimum set of measurements to exploit this freedom.

In Ref. [16], constraints on the frequency responses of the sprung mass position, the suspension travel, and the tire deflection were derived for various choices of measurements ranging from the suspension travel to a full set of state variables. These constraints typically arise in the form of finite and nonzero invariant frequency points and the growth restrictions on the frequency responses and their derivatives at zero and infinite frequencies. The quarter-car model studied in Ref. [16] does not include passive suspension elements spring and damper; and also tire damping is neglected.

In most works, tire damping is set to zero when modeling automotive active suspension systems. This is partly due to the fact that tire damping is difficult to estimate. It is generally accepted that damping ratio in a vehicle tire ranges between 0.03 and 0.10 depending on the size, applied pressure, free or rotating, new or worn, and the tire type, i.e., all season or snow [18–20]. The tire damping by itself has little influence on the wheel-hop vibration since this mode is mainly damped by the shock absorber.

The ignorance of damping in tire models compelled misleading conclusions that at the wheel-hop frequency, no matter what forces are exerted between sprung and unsprung masses, their motion are uncoupled, and the vertical acceleration of the sprung mass will be unaffected [13,14,16]. It is pointed out in Ref. [21] that by taking tire damping to be small but nonzero, the motions of the sprung and unsprung masses are coupled at all frequencies, and control forces can be used to reduce the sprung mass vertical acceleration at the wheel-hop frequency. The effect of introducing tire damping can be quite large.

The paper is structured as follows. First, the results in Ref. [16] are complemented assuming that (i) the sprung mass acceleration measurement instead of the sprung mass position measurement is used for the parametrization of the stabilizing controllers, (ii) the closed-loop sprung mass acceleration is targeted instead of the closed-loop sprung mass position for the evaluation of ride comfort, (iii) the passive suspension elements are included in the vehicle model. The differences and the similarities between the derived results and Ref. [16] are emphasized. For example, it is demonstrated that employment of the sprung mass acceleration as a measurement and performance objective in a vehicle model that includes passive suspension elements affects the parameterizability of the stabilizing controllers.

The reader is reminded that a semi-active suspension consists of in series a spring and damper whose coefficient is changed in a nonlinear fashion. In a semi-active suspension, ride comfort is taken care of by a nonlinear damper while safety requirements are met by a fixed spring.

Next, the effect of tire damping on the achievable performance is investigated. The results predicate the conclusions in Refs. [21–23] that tire damping couples the motions of the sprung and unsprung masses, and control forces can be used to reduce the sprung mass vertical acceleration at the wheel-hop frequency without sacrificing road holding.

2. The quarter-car model

A two-degree-of-freedom quarter-car model is shown in Fig. 1. In this model, the sprung and unsprung masses are denoted, respectively, by m_s and m_u. The suspension system is represented by a linear spring of stiffness k_s and a linear damper with a damping rate c_s. The tire is modeled by a linear spring of stiffness k_t and a linear damper with a damping rate c_t. The parameter values, except c_t, chosen for this study are shown in Table 1 [24]. They are typical for a lightly damped passenger car. The parameters m_s, m_u, and k_t are fixed throughout the paper while the parameters k_s, c_s, c_t are freely changed.

Assuming that the tire behaves as a point-contact follower that is in contact with the road at all times, the equations of motion take the form:

$$m_s\ddot{\mathbf{x}}_1 = -k_s(\mathbf{x}_1 - \mathbf{x}_2) - c_s(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) - u,$$

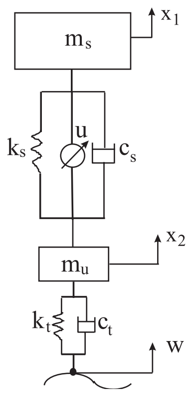

Fig. 1. The quarter-car model of the vehicle.

Table 1
The vehicle system parameters for the quarter-car model

Sprung mass.	m_s	240 kg
Unsprung mass	m_u	36 kg
Damping coefficient	c_s	980 N s m^{-1}
Secondary suspension stiffness	k_s	16,000 N m^{-1}
Primary suspension stiffness	k_t	160,000 N m^{-1}

$$m_u\ddot{\mathbf{x}}_2 = k_s(\mathbf{x}_1 - \mathbf{x}_2) + c_s(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) + u - k_t(\mathbf{x}_2 - w) - c_t(\dot{\mathbf{x}}_2 - \dot{w}), \tag{1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are, respectively, the displacements of the sprung and unsprung masses, and w is the road unevenness. The variables $\mathbf{x}_1, \mathbf{x}_2$, and w are measured with respect to an inertial frame, and the control input u is a force.

The objective of this paper is to study the performance limits of an actively controlled vehicle imposed by the road surface unevenness. The vehicle response variables that need to be examined are the vertical acceleration of the sprung mass as an indicator of the vibration isolation, the suspension travel as a measure of the rattling space, and the tire deflection as an indicator of the road-holding characteristic of the vehicle. These variables, denoted, respectively by $\mathbf{z}_1, \mathbf{z}_2$, and $\mathbf{z}_3$, can be written in terms of the state variables $\mathbf{x}_1, \mathbf{x}_2$, their derivatives, and the exogenous input w as follows:

$$\mathbf{z}_1 = \ddot{\mathbf{x}}_1, \tag{2}$$

$$\mathbf{z}_2 = \mathbf{x}_1 - \mathbf{x}_2, \tag{3}$$

$$\mathbf{z}_3 = \mathbf{x}_2 - w. \tag{4}$$

Passenger comfort requires $\mathbf{z}_1$ to be as small as possible while compactness of rattle space, good handling characteristics, and improved road-holding quality require $\mathbf{z}_2$ and $\mathbf{z}_3$ be kept as small as possible.

It is a well-known fact [15] that these objectives cannot be met simultaneously with a passive suspension system. In a passive suspension system, the only parameter that can be altered in an optimization study is c_s since k_s is *a priori* fixed to obtain stiffness against rolling. The conflicting three goals can be attained up to a certain level by replacing passive suspension system with an active or semi-active suspension system [2–4,7,8,16,24].

3. Factorization approach to feedback stability

In this section, the factorization approach developed in Ref. [16] for the feedback stability of the quarter-car model is briefly reviewed. The reader is referred to Vidyasagar and Zhou et al. [25,26] for a comprehensive treatment.

Let $\mathbf{Z}(s)$, $U(s)$, and $W(s)$ denote, respectively, the Laplace transforms of the signals $\mathbf{z}(t) = [\mathbf{z}_1(t)\,\mathbf{z}_2(t)\,\mathbf{z}_3(t)]^{\mathrm{T}}$, $u(t)$, and $w(t)$, where for a given vector $\mathbf{b}$, $\mathbf{b}^{\mathrm{T}}$ denotes the transpose of $\mathbf{b}$. From Eqs. (1)–(4),

$$\mathbf{Z}(s) = \mathbf{G}_{11}(s)W(s) + \mathbf{G}_{12}(s)U(s), \tag{5}$$

where

$$\mathbf{G}_{11}(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2(c_s s + k_s)(c_t s + k_t) \\ -m_s s^2(c_t s + k_t) \\ -s^2[m_s m_u s^2 + (m_s + m_u)c_s s + (m_s + m_u)k_s] \end{bmatrix}, \tag{6}$$

$$\mathbf{G}_{12}(s) = \frac{1}{\Delta(s)}\begin{bmatrix} -s^2(m_u s^2 + c_t s + k_t) \\ -[(m_s + m_u)s^2 + c_t s + k_t] \\ m_s s^2 \end{bmatrix} \tag{7}$$

and

$$\begin{aligned}\Delta(s) = {} & m_s m_u s^4 + [(m_s + m_u)c_s + m_s c_t]s^3 + [(m_s + m_u)k_s + m_s k_t + c_s c_t]s^2 \\ & + (c_s k_t + c_t k_s)s + k_s k_t.\end{aligned} \tag{8}$$

A polynomial $\Delta(s)$ is said to be *Hurwitz* if all its zeros lie in the open left-half plane. Note that $\Delta(s)$ is Hurwitz if $k_s, k_t > 0$, and $c_s > 0$ or $c_t > 0$.

For the design of a feedback law, consider the measurements:

$$\mathbf{y}_1 = \ddot{\mathbf{x}}_1,$$

$$\mathbf{y}_2 = \mathbf{x}_1 - \mathbf{x}_2. \tag{9}$$

In the study of the constraints, the cases $\mathbf{y} = \mathbf{y}_2$ and $\mathbf{y} = [\mathbf{y}_1\,\mathbf{y}_2]^{\mathrm{T}}$ will be considered. When $\mathbf{y} = [\mathbf{y}_1\,\mathbf{y}_2]^{\mathrm{T}}$, from Eqs. (1)–(4),

$$\mathbf{Y}(s) = \mathbf{G}_{21}(s)W(s) + \mathbf{G}_{22}(s)U(s), \tag{10}$$

where

$$\mathbf{G}_{21}(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2(c_s s + k_s)(c_t s + k_t) \\ -m_s s^2(c_t s + k_t) \end{bmatrix}, \tag{11}$$

$$\mathbf{G}_{22}(s) = -\frac{1}{\Delta(s)}\begin{bmatrix} s^2(m_u s^2 + c_t s + k_t) \\ (m_s + m_u)s^2 + c_t s + k_t \end{bmatrix}. \tag{12}$$

The other case is obtained by simply selecting the second rows of $\mathbf{G}_{21}$ and $\mathbf{G}_{22}$. Hence, the generalized plant defined by

$$\mathbf{G}(s) = \left[\begin{array}{c|c} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \hline \mathbf{G}_{21} & \mathbf{G}_{22} \end{array}\right] \tag{13}$$

maps the pair of inputs $[w\,u]^{\mathrm{T}}$ to the pair of outputs $[\mathbf{z}^{\mathrm{T}}\,\mathbf{y}^{\mathrm{T}}]^{\mathrm{T}}$.

Now, let $\mathbf{K}(s)$ denote the transfer function of the controller with input $\mathbf{y}$ and the output u. The feedback configuration is shown in Fig. 2. The stabilization problem is to find a proper feedback transfer function $\mathbf{K}$ such that the closed-loop system in Fig. 2 is *internally stable*. Assuming that $\mathbf{G}$ and $\mathbf{G}_{22}$ share the same unstable poles, it is a well-known fact (see, for example, Lemma 12.2 in Ref. [26]) that $\mathbf{K}$ internally stabilizes $\mathbf{G}$ if and

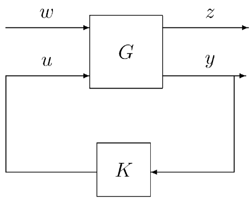

Fig. 2. Standart block diagram.

only if $\mathbf{K}$ internally stabilizes $\mathbf{G}_{22}$. Recall that the unstable poles of $\mathbf{G}$ are the roots of $\Delta(s)$ in the closed right-half plane. Assuming that $\mathbf{G}_{22}$ is internally stabilizable, the set of all compensators which stabilize $\mathbf{G}$ can be parametrized in terms of a *coprime factorization* of $\mathbf{G}_{22}$. This parametrization is called the Youla parametrization.

The Youla parametrization is obtained as follows. Let $\mathscr{RH}_\infty^{pq}$ denote the set of stable p by q real-rational transfer matrices. (In what follows, superscripts p and q will be dropped and they will be inferred from the underlying context). Given $\mathbf{G}_{22}$, find matrices $\mathbf{N}$, $\mathbf{M}$, $\widetilde{\mathbf{N}}$, $\widetilde{\mathbf{M}}$, $\mathbf{X}$, $\mathbf{Y}$, $\widetilde{\mathbf{X}}$, and $\widetilde{\mathbf{Y}}$ in $\mathscr{RH}_\infty$ such that

$$\mathbf{G}_{22} = \mathbf{N}\mathbf{M}^{-1} = \widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{N}}, \tag{14}$$

$$\begin{bmatrix} \widetilde{\mathbf{X}} & -\widetilde{\mathbf{Y}} \\ -\widetilde{\mathbf{N}} & \widetilde{\mathbf{M}} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{Y} \\ \mathbf{N} & \mathbf{X} \end{bmatrix} = \mathbf{I}, \tag{15}$$

where $\mathbf{I}$ denotes the identity matrix of compatible dimension. The factorization Eq. (14) of $\mathbf{G}_{22}$ satisfying Eq. (15) is called *double* coprime factorization over $\mathscr{RH}_\infty$. Then, the Youla parametrization of all stabilizing controllers takes the form:

$$\mathbf{K} = (\mathbf{Y} - \mathbf{M}\mathbf{Q})(\mathbf{X} - \mathbf{N}\mathbf{Q})^{-1}, \quad \mathbf{Q} \in \mathscr{RH}_\infty \det(\mathbf{I} - \mathbf{X}^{-1}\mathbf{N}\mathbf{Q})(\infty) \neq 0. \tag{16}$$

With this parametrization, the transfer matrix from w to $\mathbf{z}$ denoted by $\mathbf{T}_{\mathbf{z}w}(s)$ takes a particularly convenient form which is affine in $\mathbf{Q}$:

$$\mathbf{T}_{\mathbf{z}w} = \mathbf{G}_{11} + \mathbf{G}_{12}(\mathbf{Y} - \mathbf{M}\mathbf{Q})\widetilde{\mathbf{M}}\mathbf{G}_{21}. \tag{17}$$

As $\mathbf{Q}$ varies over $\mathscr{RH}_\infty$, Eq. (17) parametrizes all achievable transfer matrices.

4. Achievable performance for quarter car model

In the design of an active suspension system, it is desirable to keep the road response amplitudes $|\mathbf{T}_{\mathbf{z}_k w}(j\omega)|$, $k = 1, 2, 3$ as small as possible, at least in the frequency range of interest. The aim of this section is to investigate the limitations on this goal for the two measurement setups and several assumptions on k_s, c_s, and c_t. By using the factorization approach, a complete set of constraints on the transfer functions $\mathbf{T}_{\mathbf{z}_k w}(s)$, $k = 1, 2, 3$ will be derived.

The first case to be treated in the sequel is the case $\mathbf{y} = \mathbf{y}_2$ with the assumption that k_s, c_s, and c_t are all positive. Then, as noted previously, $\Delta(s)$ is a Hurwitz polynomial and a pair of coprime factors for

$$\mathbf{G}_{22} = -\Delta^{-1}[(m_s + m_u)s^2 + c_t s + k_t] \tag{18}$$

is easily found as

$$\mathbf{N} = \widetilde{\mathbf{N}} = \mathbf{G}_{22}, \qquad \mathbf{M} = \widetilde{\mathbf{M}} = 1. \tag{19}$$

Furthermore, the choice given by

$$\mathbf{X} = \widetilde{\mathbf{X}} = \mathbf{G}_{22} + 1, \qquad \mathbf{Y} = \widetilde{\mathbf{Y}} = 1 \tag{20}$$

enforces Eq. (15) as can directly be verified. Now, put $\widehat{\mathbf{Q}} = 1 - \mathbf{Q}$ where $\mathbf{Q} \in \mathcal{RH}_\infty$. Then, $\mathbf{T}_{\mathbf{z}w}$ and $\mathbf{K}$ in Eqs. (17) and (16)) take the following forms:

$$\mathbf{T}_{\mathbf{z}_1 w} = s^2 \Delta^{-1}(c_t s + k_t)\{c_s s + k_s + \widehat{\mathbf{Q}} m_s s^2 \Delta^{-1}(m_u s^2 + c_t s + k_t)\}, \tag{21}$$

$$\mathbf{T}_{\mathbf{z}_2 w} = -m_s s^2 \Delta^{-1}(c_t s + k_t)\{1 - \widehat{\mathbf{Q}} \Delta^{-1}[(m_s + m_u)s^2 + c_t s + k_t]\}, \tag{22}$$

$$\begin{aligned}\mathbf{T}_{\mathbf{z}_3 w} = & - s^2 \Delta^{-1}[m_s m_u s^2 + (m_s + m_u)c_s s + (m_s + m_u)k_s] \\ & - \widehat{\mathbf{Q}} m_s^2 s^4 \Delta^{-2}(c_t s + k_t),\end{aligned} \tag{23}$$

$$\mathbf{K} = \widehat{\mathbf{Q}}(1 + G_{22}\widehat{\mathbf{Q}})^{-1}. \tag{24}$$

The first constraint is obtained by observing that the coefficient of $\widehat{\mathbf{Q}}$ in the expression for $\mathbf{T}_{\mathbf{z}_1 w}$ is $O(s^4)$ for all sufficiently small complex numbers s and every transfer matrix in $\mathcal{RH}_\infty$ has elements uniformly bounded on the closed right half-plane. Here, the notation $a(s) = O(s^\gamma)$ means that there exist two numbers $\alpha, \beta > 0$ such that

$$\alpha |s|^\gamma \leqslant |a(s)| \leqslant \beta |s|^\gamma.$$

Therefore, a Taylor series expansion of the term $s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t)$ in a neighborhood of zero which is accurate up to the term $O(s^4)$ will be sufficient to determine the behavior of $\mathbf{T}_{\mathbf{z}_1 w}$ there. By long division,

$$s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t) = s^2 + O(s^4). \tag{25}$$

Hence, for all small s,

$$\mathbf{T}_{\mathbf{z}_1 w} = s^2 + O(s^4), \tag{26}$$

which implies,

$$\mathbf{T}_{\mathbf{z}_1 w}(0) = \mathbf{T}^{(1)}_{\mathbf{z}_1 w}(0) = \mathbf{T}^{(3)}_{\mathbf{z}_1 w}(0) = 0, \qquad \mathbf{T}^{(2)}_{\mathbf{z}_1 w}(0) = 2. \tag{27}$$

For all large s, observe that the coefficient of $\widehat{\mathbf{Q}}$ in the expression for $\mathbf{T}_{\mathbf{z}_1 w}$ is $O(s^{-1})$. Thus, a Taylor series expansion of the term $s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t)$ around infinity which is accurate up to a term $O(s^{-1})$ is obtained, again by long division, as

$$s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t) = \frac{c_s c_t}{m_s m_u} + O(s^{-1}). \tag{28}$$

Hence,

$$\mathbf{T}_{\mathbf{z}_1 w} = \frac{c_s c_t}{m_s m_u} + O(s^{-1}). \tag{29}$$

It remains to show that Eqs. (26) and (29) form a complete set of constraints, i.e., no further constraints on $\mathbf{T}_{\mathbf{z}_1 w}$, which are valid for all $\mathbf{Q} \in \mathcal{RH}_\infty$, can be derived. This amounts to showing that given an arbitrary $\mathbf{H}_1 \in \mathcal{RH}_\infty$ subject to constraints Eqs. (26) and (29), there exists a stabilizing controller such that for some $\widehat{\mathbf{Q}} \in \mathcal{RH}_\infty$, Eq. (21) holds for $\mathbf{H}_1$. In this case, $\mathbf{H}_1$ is said *admissible*. To this end, from Eq. (21)

$$\widehat{\mathbf{Q}} = \frac{[T_{z_1 w} - s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t)]}{m_s s^4 \Delta^{-2}(m_u s^2 + c_t s + k_t)(c_t s + k_t)}.$$

From Eqs. (28) and (29), the numerator and the denumerator of $\widehat{\mathbf{Q}}$ are $O(s^{-1})$. Hence, $\widehat{\mathbf{Q}}$ is a proper rational function with a singularity at the origin of multiplicity four. However, the singularity at the origin is removable from Eqs. (25) and (26). Thus, $\widehat{\mathbf{Q}} \in \mathcal{RH}_\infty$ as desired.

It should be noted that as soon as an admissible $\mathbf{H}_1$ is specified, two other admissible functions, be $\mathbf{H}_2$ and $\mathbf{H}_3$, corresponding to $\mathbf{T}_{z_2 w}$ and $\mathbf{T}_{z_3 w}$ are generated via Eqs. (22) and (23). Indeed, elimination of $\widehat{\mathbf{Q}}$ in Eqs. (21)–(23) results in the following trade-off relations:

$$\mathbf{H}_2 = -\frac{c_t s + k_t}{m_u s^2 + c_t s + k_t} + \frac{(m_s + m_u)s^2 + c_t s + k_t}{s^2(m_u s^2 + c_t s + k_t)} \mathbf{H}_1, \tag{30}$$

$$\mathbf{H}_3 = -\frac{m_u s^2}{m_u s^2 + c_t s + k_t} - \frac{m_s}{m_u s^2 + c_t s + k_t}\mathbf{H}_1, \tag{31}$$

$$\mathbf{H}_3 = -\frac{(m_s + m_u)s^2}{(m_s + m_u)s^2 + c_t s + k_t} - \frac{m_s s^2}{(m_s + m_u)s^2 + c_t s + k_t}\mathbf{H}_2. \tag{32}$$

The constraints on $\mathbf{H}_1$ and its derivatives at $s = 0$ are partially recovered by Eqs. (30) and (31). In fact, given an admissible $\mathbf{H}_2$, analyticity of $\mathbf{H}_2$ at $s = 0$ forces the following function:

$$\mathbf{H}_1(s) - s^2 \Delta^{-1}(c_s s + k_s)(c_t s + k_t)$$

to have at least two zeros there, which implies $\mathbf{H}_1(0) = \mathbf{H}_1'(0) = 0$. No results on further derivatives of $\mathbf{H}_1$ at $s = 0$ can be deduced from Eqs. (30) and (31).

The above results are captured in the following.

Proposition 4.1. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_1$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_1 = \mathbf{T}_{\mathbf{z}_1 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_1(s) = c_s c_t / m_s m_u + O(s^{-1})$,
(2) $\mathbf{H}_1(0) = \mathbf{H}_1^{(1)}(0) = \mathbf{H}_1^{(3)}(0) = 0,\ \mathbf{H}_1^{(2)}(0) = 2$.

A similar derivation to the above can be carried out for $\mathbf{H}_2$ and $\mathbf{H}_3$ or Proposition 4.1 combined with Eqs. (30) and (31) yields the following results.

Proposition 4.2. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_2$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_2 = \mathbf{T}_{\mathbf{z}_2 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_2(s) = -(c_t/m_u)s^{-1} + [(m_s + m_u/m_u)(c_s c_t/m_s m_u) + (c_t^2/m_u^2) - (k_t/m_u)]s^{-2} + O(s^{-3})$,
(2) $\mathbf{H}_2(0) = \mathbf{H}_2^{(1)}(0) = 0$.

Proposition 4.3. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_3$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_3 = \mathbf{T}_{\mathbf{z}_3 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_3(s) = -1 + (c_t/m_u)s^{-1} + [(k_t/m_u) - (c_s + c_t)c_t/m_u^2]s^{-2} + O(s^{-3})$,
(2) $\mathbf{H}_3(0) = \mathbf{H}_3^{(1)}(0) = 0,\ \mathbf{H}_3^{(2)}(0) = -2(m_s + m_u)/k_t,\ \mathbf{H}_3^{(3)}(0) = 6(m_s + m_u)c_t/k_t^2$.

Now, assume that $c_t = 0$ and $k_s, c_s > 0$. Then, $\Delta(s)$ is still a Hurwitz polynomial, and it suffices to let $c_t = 0$ in Eqs. (21)–(23) and Eqs. (30)–(32). Two new constraints arise at the frequencies:

$$\omega_1 = \sqrt{\frac{k_t}{m_s + m_u}}, \qquad \omega_2 = \sqrt{\frac{k_t}{m_u}} \tag{33}$$

which have already been observed in Refs. [14,16]. The results for this case are summarized in the following.

Proposition 4.4. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, *and* $c_t = 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_1$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_1 = \mathbf{T}_{\mathbf{z}_1 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_1(s) = (k_t c_s/m_s m_u)s^{-1} + O(s^{-2})$,
(2) $\mathbf{H}_1(0) = \mathbf{H}_1^{(1)}(0) = \mathbf{H}_1^{(3)}(0) = 0,\ \mathbf{H}_1^{(2)}(0) = 2$,
(3) $\mathbf{H}_1(\mathrm{j}\omega_2) = -(\mathrm{j}\omega_2)^2(m_u/m_s)$.

Proposition 4.5. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, *and* $c_t = 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_2$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_2 = \mathbf{T}_{\mathbf{z}_2 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_2(s) = -(k_t/m_u)s^{-2} + ((m_s + m_u)c_s k_t/m_s m_u^2)s^{-3} + O(s^{-4})$,
(2) $\mathbf{H}_2(0) = \mathbf{H}_2^{(1)}(0) = 0$,
(3) $\mathbf{H}_2(\mathrm{j}\omega_1) = -m_s + m_u/m_s$.

Proposition 4.6. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, *and* $c_t = 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_3$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_3 = \mathbf{T}_{\mathbf{z}_3 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_3(s) = -1 + (k_t/m_u)s^{-2} - (k_t c_s/m_u^2)s^{-3} + O(s^{-4})$,
(2) $\mathbf{H}_3(0) = \mathbf{H}_3^{(1)}(0) = \mathbf{H}_3^{(3)}(0) = 0$, $\mathbf{H}_3^{(2)}(0) = -2(m_s + m_u)/k_t$.

Propositions 4.4–4.6 yield Theorems 1–3 in Ref. [16] when $c_s = 0$. In Ref. [16], $\mathbf{T}_{\mathbf{x}_1 w}$ is constrained instead of $\mathbf{T}_{\mathbf{z}_1 w}$. The latter is related to the former by the equation $\mathbf{T}_{\mathbf{z}_1 w}(s) = s^2 \mathbf{T}_{\mathbf{x}_1 w}(s)$. Then, the third formula in Proposition 4.4 recovers the constraint $\mathbf{T}_{\mathbf{x}_1 w}(\mathrm{j}\omega_2) = -m_u/m_s$ derived in Ref. [16].

The appearance of c_s in the constraints of Propositions 4.4–4.6 demonstrates that the damper in Fig. 1 cannot be incorporated to u. Otherwise, for a given proper controller $\mathbf{K}^\sharp$ that stabilizes the quarter-car model in Fig. 1 with $c_s = k_s = 0$, a controller $\mathbf{K}$ satisfying Eq. (24) and $\mathbf{K}^\sharp = \mathbf{K} + sc_s + k_s$ would be improper since both controllers have the same input $\mathbf{y}_2$. The same argument also explains absence of k_s in the constraints of Propositions 4.1–4.6. The reader is cautioned that the first conclusion drawn above is valid for the quarter-car model with a suspension consisting of an actuator in parallel with a spring and a damper as shown in Fig. 1. There are many possibilities to connect passive elements with an actuator, in which the issue of properness never arises. In hardware implementation of active or semi-active suspensions, parallel connection (without damper) is a preferred configuration.

Although k_s does not appear in the constraint formulae above, a given set of measurements may not be sufficient to parametrize all stabilizing proper controllers if the spring in Fig. 1 is missing. Recall the internal stabilizability condition: $\mathbf{G}$ and $\mathbf{G}_{22}$ share the same unstable poles. If $c_s > 0$ or $c_t > 0$ and $k_s > 0$, then this requirement is satisfied by all elements of $\mathbf{G}_{22}$ in Eq. (12) and one can also use $\mathbf{y}_1$ for the parametrization of the stabilizing controllers. However, in general, different measurements lead to different constraint sets. If $c_s = c_t = k_s = 0$, then Δ and $\mathbf{G}_{22}$ in Eqs. (8) and (12) equal $m_s s^2(m_u s^2 + k_t)$ and

$$\begin{bmatrix} -\dfrac{1}{m_s} \\ -\dfrac{(m_s + m_u)s^2 + k_t}{m_s s^2(m_u s^2 + k_t)} \end{bmatrix}.$$

Clearly, $\mathbf{y}_1$ cannot give rise to a parametrization of stabilizing controllers. On the other hand, If $k_s > 0$, then $\Delta(s) = m_s m_u s^4 + [(m_s + m_u)k_s + m_s k_t]s^2 + k_s k_t$, and since $\Delta(\mathrm{j}\omega_1) \neq 0$ and $\Delta(\mathrm{j}\omega_2) \neq 0$, no pole-zero cancellation can happen between Δ and any component of $\mathbf{G}_{22}$. Hence, as a measurement, $\ddot{\mathbf{x}}_1$ or $\mathbf{x}_1 - \mathbf{x}_2$ is sufficient for the parametrization of the stabilizing controllers.

The next case to be studied is $\mathbf{y} = [\mathbf{y}_1\, \mathbf{y}_2]^\mathrm{T}$. Let $\widehat{\mathbf{Q}} = [\widehat{\mathbf{Q}}_1\, \widehat{\mathbf{Q}}_2] = (\mathbf{Y} - \mathbf{M}\mathbf{Q})\widetilde{\mathbf{M}}$. Since $\mathbf{Q} \in \mathscr{RH}_\infty$ is a two-dimensional row vector, $\mathbf{T}_{\mathbf{z}w}$ in Eq. (17) can be written as

$$\mathbf{T}_{\mathbf{z}w} = \mathbf{G}_{11} + \mathbf{G}_{12}\mathbf{G}_{21}^\mathrm{T}\widehat{\mathbf{Q}}^\mathrm{T}. \tag{34}$$

Recall that a non-singular matrix is *unimodular* if its determinant is constant. Now, define a product of unimodular matrices by

$$\mathbf{\Pi} = \begin{bmatrix} 1 & 0 \\ m_s^{-1} c_s s & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ m_s^{-1} k_s & 1 \end{bmatrix} \tag{35}$$

which is a unimodular matrix with the inverse:

$$\mathbf{\Pi}^{-1} = \begin{bmatrix} 1 & 0 \\ -m_s^{-1}(c_s s + k_s) & 1 \end{bmatrix}.$$

The chosen matrix $\mathbf{\Pi}$ has the property:

$$\mathbf{G}_{21}^{\mathrm{T}}\mathbf{\Pi} = -m_s s^2 \Delta^{-1}(c_t s + k_t)\,[0\;1]. \tag{36}$$

It should be clear how to proceed in order to parametrize the stabilizing controllers if more than two measurements are available for feedback. For example, if $\mathbf{y}$ equals $[\mathbf{y}_1\ \mathbf{y}_2\ \ddot{\mathbf{x}}_2\ \mathbf{x}_2]^{\mathrm{T}}$, a unimodular matrix $\mathbf{\Pi}$ is constructed such that when premultiplied with $\mathbf{G}_{21}^{\mathrm{T}}$, the result is a row vector with the first three elements being zero, similar to Eq. (36). Therefore, the two factors of $\mathbf{\Pi}$ in Eq. (35) are exactly elementary column operators. The utility of Eq. (36) is to allow a parametrization of $\mathbf{T}_{\mathbf{z}w}$ in terms of a scalar transfer function.

Since Δ is Hurwitz, coprime factors of $\mathbf{G}_{22}$ can be chosen as follows:

$$\mathbf{N} = \widetilde{\mathbf{N}} = \mathbf{G}_{22}, \qquad \mathbf{M} = 1, \qquad \widetilde{\mathbf{M}} = \mathbf{I}.$$

Hence,

$$\mathbf{T}_{\mathbf{z}w} = \mathbf{G}_{11} + \widetilde{q}\,\mathbf{G}_{12} s^2 \Delta^{-1}(c_t s + k_t)(c_s s + k_s),$$

where

$$\widetilde{q} = \widehat{\mathbf{Q}}_1 - \widehat{\mathbf{Q}}_2 m_s (c_s s + k_s)^{-1}. \tag{37}$$

It follows that

$$\mathbf{T}_{\mathbf{z}_1 w} = s^2 \Delta^{-1}(c_t s + k_t)(c_s s + k_s)\left\{1 - \widetilde{q} s^2 \Delta^{-1}(m_u s^2 + c_t s + k_t)\right\}, \tag{38}$$

$$\mathbf{T}_{\mathbf{z}_2 w} = -s^2 \Delta^{-1}(c_t s + k_t) \cdot \left\{m_s + \widetilde{q}\Delta^{-1}[(m_s + m_u)s^2 + c_t s + k_t](c_s s + k_s)\right\}, \tag{39}$$

$$\begin{aligned}\mathbf{T}_{\mathbf{z}_3 w} = &-s^2 \Delta^{-1}[m_s m_u s^2 + (m_s + m_u)c_s s + (m_s + m_u)k_s] \\ &+ \widetilde{q}\, m_s s^4 \Delta^{-2}(c_s s + k_s)(c_t s + k_t),\end{aligned} \tag{40}$$

$$\mathbf{K} = \widehat{\mathbf{Q}}(\mathbf{I} + \mathbf{G}_{22}\widehat{\mathbf{Q}})^{-1}. \tag{41}$$

The range space of $\widetilde{q}$ equals $\mathscr{RH}_\infty$. This means that $\widetilde{q}$ and $\widehat{\mathbf{Q}}_2$ can be used to parametrize the set of the stabilizing controller. Then, $\widehat{\mathbf{Q}}_1$ is solved from Eq. (37) and plugged in Eq. (41).

The trade-off relations among $\mathbf{H}_1, \mathbf{H}_2$, and $\mathbf{H}_3$ are the same as Eqs. (30)–(32). The constraints on the transfer functions $\mathbf{H}_1, \mathbf{H}_2$, and $\mathbf{H}_3$ are expressed in the following results.

Proposition 4.7. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ *and let* $\mathbf{H}_1$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_1 = \mathbf{T}_{\mathbf{z}_1 w}$ *for some stabilizing control law if and only if*:

$$\mathbf{H}_1(0) = \mathbf{H}_1^{(1)}(0) = \mathbf{H}_1^{(3)}(0) = 0,\ \mathbf{H}_1^{(2)}(0) = 2.$$

Proposition 4.8. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = \mathbf{y}_2$ *and let* $\mathbf{H}_2$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_2 = \mathbf{T}_{\mathbf{z}_2 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_2(s) = (-c_t/m_u)s^{-1} + O(s^{-2})$,
(2) $\mathbf{H}_2(0) = \mathbf{H}_2^{(1)}(0) = 0$.

Proposition 4.9. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s, c_t > 0$. *Assume that* $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ *and let* $\mathbf{H}_3$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_3 = \mathbf{T}_{\mathbf{z}_3 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_3(s) = -1 + (c_t/m_u)s^{-1} + O(s^{-2})$,
(2) $\mathbf{H}_3(0) = \mathbf{H}_3^{(1)}(0) = 0$, $\mathbf{H}_3^{(2)}(0) = -2(m_s + m_u)/k_t$, $\mathbf{H}_3^{(3)}(0) = 6(m_s + m_u)c_t/k_t^2$.

Based on Propositions 4.1–4.3 and Propositions 4.7–4.9, the following conclusions can be drawn. First, extra measurement, i.e., $\mathbf{y}_1$ affects only the Taylor series coefficients of $\mathbf{H}_1, \mathbf{H}_2$, and $\mathbf{H}_3$ at infinity. Second, c_s does not show up in the formulae of Propositions 4.7–4.9 in contrast to those of Propositions 4.1–4.3. Finally, the trade-off relations are the same for both cases. This implies that a point in the $\mathbf{H}_2$ versus $\mathbf{H}_1$ trade-off curve determined uniquely by the controller in Eq. (24) corresponds to an infinite number of controllers in Eq. (41). This information can be useful in constraining the controller dynamics. For example, constraining the ℓ_1 norm of $\mathbf{K}$, which is defined as the absolute integral of the impulse response of $\mathbf{K}$, constrains the magnitude of the input to persistent measurements.

The last case to be studied in this paper is the case $k_s, c_s > 0$, $c_t = 0$, and $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$. As noted before, Δ is Hurwitz, and it suffices to let $c_t = 0$ in Eqs. (38)–(40). The trade-off relations are the same as Eqs. (30)–(32) with $c_t = 0$ substituted. The constraints on $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ are captured in the following results.

Proposition 4.10. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, $c_t = 0$. *Assume that* $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ *and let* $\mathbf{H}_1$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_1 = \mathbf{T}_{\mathbf{z}_1 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_1(s) = O(s^{-1})$,
(2) $\mathbf{H}_1(0) = \mathbf{H}_1^{(1)}(0) = \mathbf{H}_1^{(3)}(0) = 0,\ \mathbf{H}_1^{(2)}(0) = 2$,
(3) $\mathbf{H}_1(\mathrm{j}\omega_2) = -(\mathrm{j}\omega_2)^2 m_u/m_s$.

Proposition 4.11. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, *and* $c_t = 0$. *Assume that* $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ *and let* $\mathbf{H}_2$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_2 = \mathbf{T}_{\mathbf{z}_2 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_2(s) = -(k_t/m_u)s^{-2} + O(s^{-3})$,
(2) $\mathbf{H}_2(0) = \mathbf{H}_2^{(1)}(0) = 0$,
(3) $\mathbf{H}_2(\mathrm{j}\omega_1) = -m_s + m_u/m_s$.

Proposition 4.12. *Consider the quarter car model in Eq.* (1) *with* $k_s, c_s > 0$, $c_t = 0$. *Assume that* $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ *and let* $\mathbf{H}_3$ *be any function in* $\mathscr{RH}_\infty$. *Then,* $\mathbf{H}_3 = \mathbf{T}_{\mathbf{z}_3 w}$ *for some stabilizing control law if and only if*:

(1) $\mathbf{H}_3(s) = -1 + (k_t/m_u)s^{-2} + O(s^{-3})$,
(2) $\mathbf{H}_3(0) = \mathbf{H}_3^{(1)}(0) = \mathbf{H}_3^{(3)}(0) = 0,\ \mathbf{H}_3^{(2)}(0) = -2(m_s + m_u)/k_t$.

As in Propositions 4.7–4.9,, c_s does not show up in the above formulae. Thus, whenever $\mathbf{y} = [\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$, the damper can be modeled as part of the actuator without affecting the interpolation conditions. The last result is in contrast with the single measurement case of Propositions 4.1–4.6. When $c_s = c_t = 0$, Δ needs not be Hurwitz; but a slightly more complicated coprime factorization can be performed easily. The reader is referred to Ref. [16] for more details on this.

The question of controller approximation is in order. More specifically, let $\mathbf{K}$ and $\bar{\mathbf{K}}$ be two stabilizing controllers obtained for the quarter-car model with the measurements $\mathbf{y}_2$ and $[\mathbf{y}_1\ \mathbf{y}_2]^{\mathrm{T}}$ and $\mathbf{H}_k$ and $\bar{\mathbf{H}}_k$ for $k = 1, 2, 3$ denote the corresponding closed-loop transfer functions, respectively. In Ref. [16], when k_s, c_s, and c_t are all zero, it is shown that the closed-loop transfer functions obtained with a stabilizing controller that uses the measurements $\mathbf{y}_2, \dot{\mathbf{x}}_1, \mathbf{z}_3$, and $\dot{\mathbf{x}}_2$ can be approximated within a specified tolerance by the closed-loop transfer functions of a stabilizing controller that uses only the suspension travel measurement.

From Eqs. (21) and (38),

$$\mathbf{H}_1 - \bar{\mathbf{H}}_1 = (c_t s + k_t)s^4 \Delta^{-2}(m_u s^2 + c_t s + k_t)\,[m_s \widehat{Q} + \widetilde{q}(c_s s + k_s)]. \tag{42}$$

Hence,

$$\mathbf{H}_1(\infty) - \bar{\mathbf{H}}_1(\infty) = \frac{c_s c_t}{m_s^2 m_u}\,\widetilde{q}(\infty).$$

If $\widetilde{q}(\infty)\neq 0$ and $c_t>0$, there is no way of arbitrarily well approximating $\bar{\mathbf{K}}$ by a stabilizing compensator $\mathbf{K}$ for the entire range of frequencies. However, if $\widetilde{q}(\infty)=0$, i.e., when $\widetilde{q}(s)$ is a strictly proper transfer function, by setting $\widehat{Q}=-\widetilde{q}(sc_s+k_s)m_s^{-1}$, we get $\mathbf{H}_1(s)\equiv\bar{\mathbf{H}}_1(s)$ whether c_t equals zero or not. Note from Eq. (37) that $\widetilde{q}(\infty)=0$ if and only if $\widehat{Q}_1(\infty)=0$. The latter equality does not require $\bar{\mathbf{K}}$ to be strictly proper.

Now, consider the case $c_t=0$. Let

$$\widehat{Q}=-\frac{\widetilde{q}(sc_s+k_s)}{m_s(1+\varepsilon s)}\qquad(\varepsilon>0).$$

As $\varepsilon\to 0$, $m_s\widehat{Q}+\widetilde{q}(sc_s+k_s)\to 0$ uniformly on every internal $[0,\mathrm{j}\lambda)$, $\lambda>0$ though outside this interval it diverges as $O(s)$. However, outside the internal the growth is controlled by the factor $k_ts^4(m_us^2+k_t)\Delta^{-2}=O(s^{-2})$. Picking λ sufficiently large and ε sufficiently small, the left-hand side of Eq. (42) can be made as small as desired. The convergences $\mathbf{H}_2\to\bar{\mathbf{H}}_2$ and $\mathbf{H}_3\to\bar{\mathbf{H}}_3$ as $\varepsilon\to 0$ follow from the fact that $\mathbf{H}_2$ and $\mathbf{H}_3$ are continuous functions of $\mathbf{H}_1$ on the closed right half-plane. The controller approximation result is captured in the following.

Proposition 4.13. *Let* $\bar{\mathbf{H}}_k$, $k=1,2,3$ *be the closed-loop transfer functions obtained for the quarter car model in Eq.* (1) *with* $k_s,c_s>0$, *the measurements* $\mathbf{y}_1,\mathbf{y}_2$, *and some stabilizing controller* $\bar{\mathbf{K}}$. *If* $c_t=0$ *or* $\widetilde{q}(\infty)=0$, *then for each* $\varepsilon>0$ *a stabilizing controller* $\mathbf{K}$ *that uses only* $\mathbf{y}_2$ *can be found with the corresponding transfer functions* $\mathbf{H}_k$ *satisfying* $|\bar{\mathbf{H}}_k(\mathrm{j}\omega)-\mathbf{H}_k(\mathrm{j}\omega)|<\varepsilon$ *for all* ω *and* $k=1,2,3$.

When $c_t=0$, from Propositions 4.6 and 4.12 we have $\mathbf{T}_{\mathbf{z}_3w}(s)=-1+O(s^{-2})$ for all large s, and in this case Theorem 4 in Ref. [16] reads out

$$\int_0^\infty \ln|\mathbf{T}_{\mathbf{z}_3w}(\mathrm{j}\omega)|\,\mathrm{d}\omega=\pi\sum_{k=1}^n\sigma_k,$$

where σ_k, $k=1,\dots,n$ denote the real parts of the zeros of $\mathbf{T}_{\mathbf{z}_3w}$ in the open right half-plane. This is not a quantitative but a qualitative statement expressing the difficulty of controlling the tire deflection on a broad band of frequencies due to the presence of nonminimum phase zeros.

From Propositions 4.1–4.13, the following conclusions can be drawn:

- When tire damping is present, utilizing the sprung mass acceleration measurement in addition to the suspension travel measurement in the feedback law affects only the curvatures of $\mathbf{T}_{\mathbf{z}_2w}$ and $\mathbf{T}_{\mathbf{z}_3w}$ at infinity, and $\mathbf{T}_{\mathbf{z}_1w}(\infty)$. If tire damping is neglected, one degree higher order terms of the Taylor series expansions of $\mathbf{T}_{\mathbf{z}_kw}$, $k=1,2,3$ at infinity are affected by the additional measurement.
- Closed-loop performance of any stabilizing feedback law which uses the sprung mass acceleration and the suspension travel measurements can be obtained within an arbitrary precision by a stabilizing feedback law relying only on the suspension travel measurement provided that either tire damping is neglected or the actuator transfer function satisfies a mild condition in the steady state.
- No matter how small, tire damping couples the wheel-hop and the heave modes. This coupling eliminates the constraints of the conventional quarter-car model, which neglects tire damping at the *so-called* invariant frequencies ω_1 and ω_2. As will be seen in the next section, tire damping improves ride comfort without sacrificing road holding.
- When the suspension travel is the only available measurement, c_s influences $\mathbf{T}_{\mathbf{z}_1w}(\infty)$, and the second-order terms of the Taylor series expansions of $\mathbf{T}_{\mathbf{z}_2w}$ and $\mathbf{T}_{\mathbf{z}_3w}$ at infinity if $c_t>0$. If tire damping is neglected, one degree higher-order terms of the Taylor series expansions of $\mathbf{T}_{\mathbf{z}_kw}$, $k=1,2,3$ at infinity are affected by c_s.
- If the measurements $\mathbf{y}_1$ and $\mathbf{y}_2$ are both used in the parametrization of the stabilizing controllers, then there is no need to consider c_s separately since it can be included in the feedback law.

The analysis of this paper and the results in Ref. [16] show that the constraints on the closed-loop transfer functions depend on the system parameters as well as the measurements. In the present work, $\ddot{\mathbf{x}}_1$ is taken as a measured signal instead of $\mathbf{x}_1$ and $\dot{\mathbf{x}}_1$ since in practice, the acceleration is measured, and the velocity and the position are constructed from the former by integration. The sprung mass acceleration measurement rather than velocity or position was also considered in Refs. [30, Section 4.5.2]. However, the parameterizability for the stabilizing controllers and the constraints on the closed-loop transfer function, in general, depend on which signal is being used. If $\mathbf{x}_1$ or $\mathbf{x}_2$ is used for the controller parameterization, then one also has to take into the filtering constraints and trade-offs [29], which are beyond the scope of the current work.

The constraints derived in this paper can be used for the purpose of comparing closed-loop performance of a proposed controller with benchmark values at specific frequencies. They do not give much information about the design of an actual controller, which is an involved process, and besides the road disturbance responses many other factors such as the load disturbance responses and the robustness issues have also to be taken into account. In Ref. [16], the constraints on the load response functions are derived. Further applications of the controller parameterization to vehicle active suspension design are reported in Refs. [31,32].

In passing, the constraints in Propositions 4.1 and 4.7 can be viewed as the interpolation constraints. Then, the problem of finding all stabilizing controller can be cast into a partial realization problem. Many variants of this formulation, i.e., interpolation with metric and minimal complexity constraints have been considered in the literature on rational interpolation [27].

5. Active control of the quarter-car model

The purpose of this section is to illustrate the effect of tire damping on the controller design for the quarter-car model in Fig. 1. The vehicle is assumed to traverse a random road profile with a constant forward velocity v. Then, the derivative of $w(t)$ is a random process denoted by $V_i(t)$.

It will be more convenient to define a new set of state variables in terms of the old state variables in Eq. (1) as follows:

$$\tilde{\mathbf{x}}_1 = \mathbf{x}_1 - \mathbf{x}_2, \qquad \tilde{\mathbf{x}}_2 = \mathbf{x}_2 - w, \qquad \tilde{\mathbf{x}}_3 = \mathbf{x}_3, \qquad \widetilde{\mathbf{x}}_4 = \mathbf{x}_4. \tag{43}$$

Thus, $\dot{\tilde{\mathbf{x}}}_1 = \mathbf{x}_3 - \mathbf{x}_4$, $\dot{\tilde{\mathbf{x}}}_2 = \mathbf{x}_4 - V_i$, and from 1, (2)–(4), (9), (43),

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\widetilde{\mathbf{x}} + \mathbf{B}_1 V_i + \mathbf{B}_2 u,$$

$$\mathbf{z} = \mathbf{C}_1\widetilde{\mathbf{x}} + \mathbf{D}_{12}u,$$

$$\mathbf{y} = \mathbf{C}_2\widetilde{\mathbf{x}} + \mathbf{D}_{22}u + \theta,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_s}{m_s} & 0 & -\dfrac{c_s}{m_s} & \dfrac{c_s}{m_s} \\ \dfrac{k_s}{m_u} & -\dfrac{k_t}{m_u} & \dfrac{c_s}{m_u} & -\dfrac{c_s + c_t}{m_u} \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \dfrac{c_t}{m_u} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{1}{m_s} \\ \dfrac{1}{m_u} \end{bmatrix},$$

$$\mathbf{C}_1 = \begin{bmatrix} -\dfrac{k_s}{m_s} & 0 & -\dfrac{c_s}{m_s} & \dfrac{c_s}{m_s} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{12} = \begin{bmatrix} -\dfrac{1}{m_s} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{C}_2 = \begin{bmatrix} -\dfrac{k_s}{m_s} & 0 & -\dfrac{c_s}{m_s} & \dfrac{c_s}{m_s} \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{22} = \begin{bmatrix} -\dfrac{1}{m_s} \\ 0 \end{bmatrix}$$

and θ is an artificially introduced zero-mean white sensor noise uncorrelated with V_i. Its covariance function denoted by $\mathbf{R}_\theta$ satisfies

$$\mathbf{R}_\theta(\tau) = \mu \mathbf{I} \delta(\tau).$$

Here, $\mu > 0$ is a design variable and $\delta(\tau)$ is the unit impulse function.

For simplicity, the random process V_i is modeled as

$$V_i = 2\pi n_0 \sqrt{\kappa v} \eta(t), \quad (44)$$

where $\eta(t)$ is a zero-mean white noise process satisfying $R_\eta(\tau) = \delta(\tau)$; and κ, n_0 are the road roughness parameters [28]. In that work, more general road profile models than the integrated white-noise model defined in Eq. (44) are discussed; and the consequences of the road profile modeling on the random vibration characteristics of the quarter-car model are studied in detail. The roughness parameters in the current study are set from Ref. [28] as $n_0 = 0.15708$ cycles per meter and $\kappa = 0.76 \times 10^{-5}$. Note the relation $\mathbf{T}_{\mathbf{z}V_i} = s^{-1}\mathbf{T}_{\mathbf{z}w}$. Thus, the $\mathbf{Q}$-parametrization of $\mathbf{T}_{\mathbf{z}V_i}$ can be deduced from the $\mathbf{Q}$-parametrization of $\mathbf{T}_{\mathbf{z}w}$. In particular, they share the same invariant frequencies ω_1 and ω_2.

The controller will be designed using the linear-quadratic-Gaussian (LQG) design methodology. Accordingly, $u(t)$ is computed by minimizing

$$J_{\text{LQG}} = \lim_{t_f \to \infty} \mathscr{E}\left\{ \int_0^{t_f} \left(\sum_{k=1}^{3} \rho_k^{-2} \mathbf{z}_k^2 + \rho_u u^2 \right) \mathrm{d}t \right\}, \quad (45)$$

where $\mathscr{E}(c)$ denotes the expected value of a given random variable c, and ρ_k, ρ_u are nonnegative weights to be chosen by the designer. In the simulation, ρ_u and ρ_k were set, respectively, equal to zero and the root-mean-square (RMS) values of the open-loop $\mathbf{z}_k$ denoted by $\text{RMS}_{\mathbf{z}_k}$. Note that even if ρ_u were set zero, the control effort is still penalized in Eq. (45) through the term $\rho_1^{-2}\mathbf{z}_1^2$.

In Figs. 3–5, the frequency response magnitudes of the passive and the active suspensions using either $\mathbf{y}_2$ only or $\mathbf{y}_1$ and $\mathbf{y}_2$ both as measurements are plotted for the parameter values in Table 1, $\mu = 10^{-8}$, and $c_t = 0$. The RMS values of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ were computed, respectively, as follows: 0.5424, 0.0046, 0.0017 (the passive suspension); 0.5240, 0.0034, 0.0016 (the active suspension with $\mathbf{y}_2$ measured); 0.5234, 0.0034, 0.0016 (the active suspension with $\mathbf{y}_1$ and $\mathbf{y}_2$ measured). The frequency responses of the active suspensions for the two measurement cases are almost identical; and thus, confirming the results in Proposition 4.13 and [16,13].

The natural frequency and the damping ratio of the heave mode are computed as $w_n^h = 1.2507$ Hz and $\zeta_1^h = 0.2178$ for the passive suspension. For the wheel-hop mode, they are computed as $w_n^{wh} = 11.0247$ Hz and $\zeta_1^{wh} = 0.2013$. The invariant frequencies are calculated from Eq. (33) as $\omega_1 = 3.832$ Hz and $\omega_2 = 10.610$ Hz. Since $\omega_2 \approx w_n^{wh}$, it is difficult to control the wheel-hop mode as clearly seen from Figs. 3 to 5. The 3.5% drop in the RMS vertical acceleration comes from the suppression of the heave mode vibration. This is possible

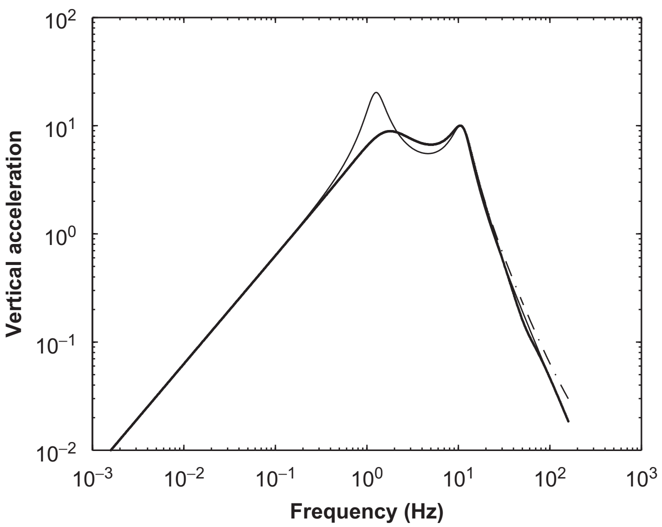

Fig. 3. The acceleration frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement without tire damping; (–.) active suspension using the acceleration and the suspension travel measurements without tire damping.

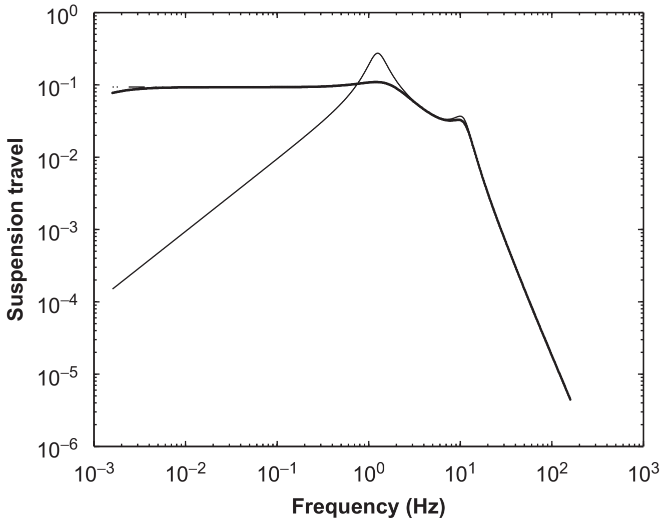

Fig. 4. The suspension travel frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement without tire damping; (—) active suspension using the acceleration and the suspension travel measurements without tire damping.

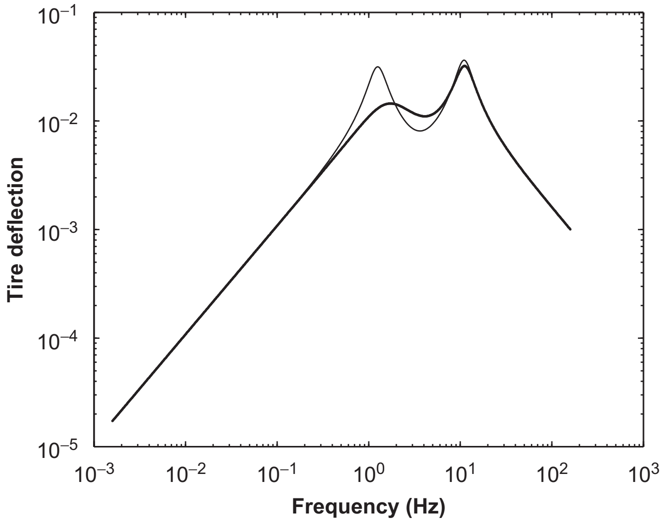

Fig. 5. The tire deflection frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement without tire damping; (–.) active suspension using the acceleration and the suspension travel measurements without tire damping.

since the natural frequency of the heave mode given approximately by $\sqrt{k_s/(m_s+m_u)}$ is well separated from ω_1.

Now, let $c_t = 2c_s$. This value is unrealistic for tire damping because it yields $w_n^h = 1.2463\,\text{Hz}$, $\zeta_1^h = 0.2211$; $w_n^{wh} = 11.0628\,\text{Hz}$, and $\zeta_1^{wh} = 0.5919$. If c_t is set to $0.1c_s$, then $w_n^h = 1.2504\,\text{Hz}$, $\zeta_1^h = 0.2180$; $w_n^{wh} = 11.0267\,\text{Hz}$, and $\zeta_1^{wh} = 0.2209$. Hence, the latter seems to be a realistic assumption. In Figs. 6–8, the counter parts of Figs. 3–5 for the same values of the vehicle and the control design parameters but $c_t = 2c_s$ are plotted. Clearly, all the three responses have been improved due to the removal of the invariant frequency at ω_2. For the RMS values of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, the following were, respectively, computed: 0.4513, 0.0043, 0.0011 (the passive suspension); 0.2834, 0.0036, 0.0010 (the active suspension with $\mathbf{y}_2$ measured); 0.2724, 0.0037, 0.0010 (the active suspension

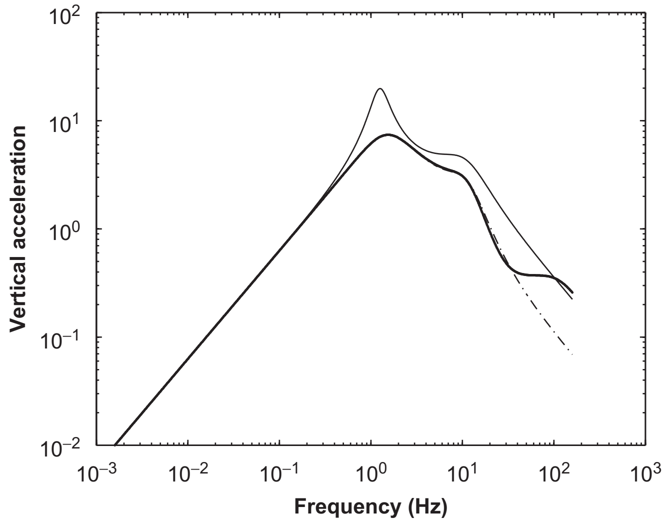

Fig. 6. The acceleration frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement with tire damping $c_t = 2c_s$: (–.) active suspension using the acceleration and the suspension travel measurements with tire damping $c_t = 2c_s$.

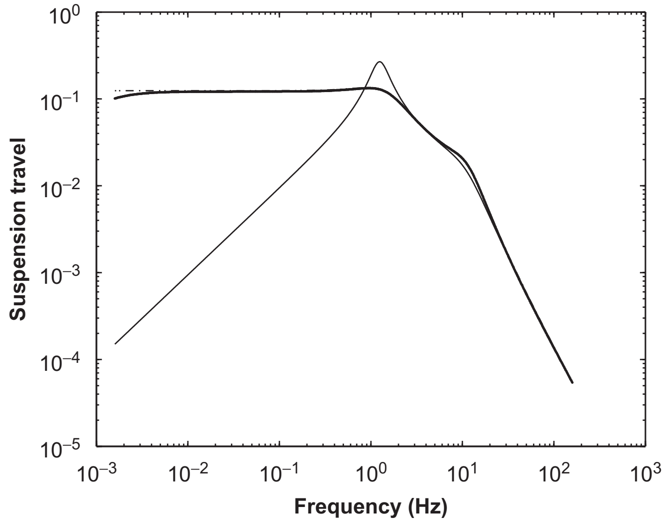

Fig. 7. The suspension travel frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement with tire damping $c_t = 2c_s$; (–.) active suspension using the acceleration and the suspension travel measurements with tire damping $c_t = 2c_s$.

with $\mathbf{y}_1$ and $\mathbf{y}_2$ measured). Comparison of Figs. 6–8 with Figs. 3–5, and the modal natural frequencies and the damping ratios shows that the improved responses are achieved by suppressing the wheel-hop vibration.

In Fig. 9, the vertical acceleration frequency response magnitude is plotted for the case $c_t = 0.1c_s$. The suspension travel and the tire deflection responses are similar to those in Figs. 4 and 5. The RMS values for this case are, respectively, 0.5259, 0.0045, 0.0017 (the passive suspension); 0.4895, 0.0034, 0.0016 (the active suspension with $\mathbf{y}_2$ measured); 0.4900, 0.0034, 0.0016 (the active suspension with $\mathbf{y}_1$ and $\mathbf{y}_2$ measured). The RMS vertical acceleration is reduced by 6.83% which is about twice of the reduction computed for the case $c_t = 0$. Though as not impressive as the overdamped tire case, the last result shows that the influence of tire damping certainly needs to be taken into account in the design of active suspensions to improve ride quality.

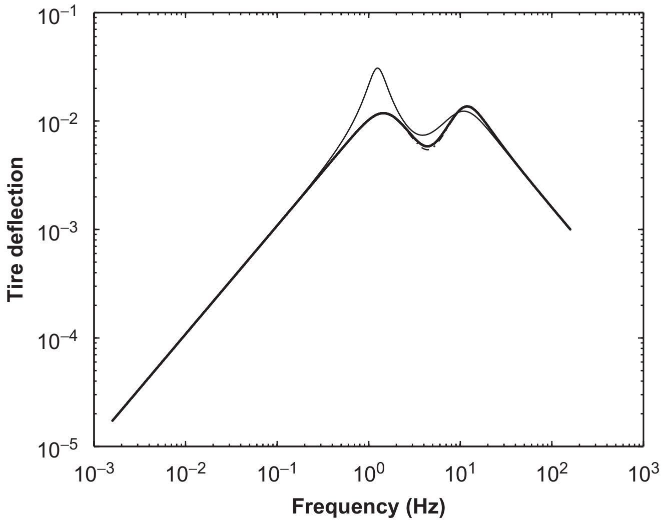

Fig. 8. The tire deflection frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement with tire damping $c_t = 2c_s$; (–.) active suspension using the acceleration and the suspension travel measurements with tire damping $c_t = 2c_s$.

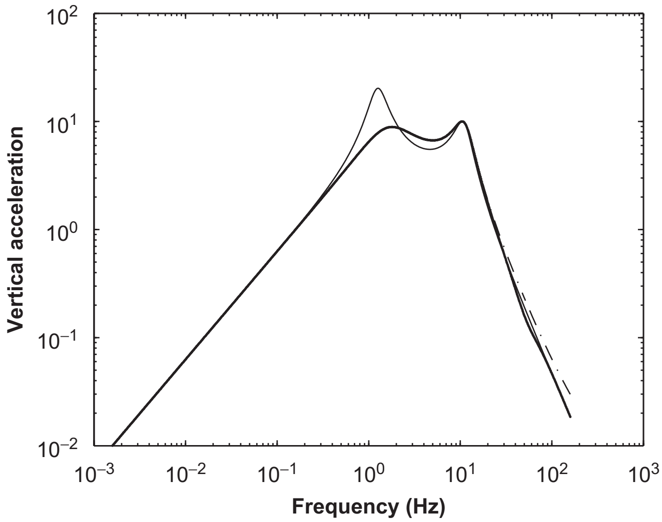

Fig. 9. The acceleration frequency response magnitude: (—) passive suspension; (...) active suspension using the suspension travel measurement with tire damping $c_t = 0.1c_s$; (–.) active suspension using the acceleration and the suspension travel measurements with tire damping $c_t = 0.1c_s$.

The rest of this section will be devoted to further enhancement of the closed-loop performance by means of the interpolation approach of this paper.

In light of the controller approximation result, it is enough to consider the case $\mathbf{y} = \mathbf{y}_2$. Recall that $\mathbf{T}_{\mathbf{z}V_i} = s^{-1}\mathbf{T}_{\mathbf{z}w}$ which implies from Eq. (21) that

$$\mathbf{T}_{\mathbf{z}_1 V_i} = s\Delta^{-1}(c_t s + k_t)\{c_s s + k_s + \widehat{\mathbf{Q}} m_s s^2 \Delta^{-1}(m_u s^2 + c_t s + k_t)\}. \tag{46}$$

Put $c_t = \alpha c_s$ ($\alpha > 0$) and $\mathbf{H}_k(s; \alpha, \widehat{\mathbf{Q}}) = \mathbf{T}_{\mathbf{z}_k V_i}$, $k = 1, 2, 3$, where we have made the dependence on the parameters c_t and $\widehat{\mathbf{Q}}$ explicit. Let $\alpha_1 = 0.1$, $\alpha_2 = 2$, and $\mathbf{Q}^\dagger$ and $\mathbf{Q}^\sharp$ denote the $\widehat{\mathbf{Q}}$ parameters of the compensators designed by the above LQG method with $c_t = \alpha_1 c_s$ and $c_t = \alpha_2 c_s$, respectively. As far as the closed-loop responses are concerned, $\mathbf{H}_k(s;\alpha_2,\mathbf{Q}^\sharp)$, $k=1,2,3$ are satisfactory while $\mathbf{H}_k(s;\alpha_1,\mathbf{Q}^\dagger)$ are not. Thus, the interpolation problem to be studied is formulated as follows:

Does there exist a $\widehat{\mathbf{Q}} \in \mathscr{R}H_\infty$ *such that* $\mathbf{H}_1(s;\alpha_1,\widehat{\mathbf{Q}}) = \mathbf{H}_1(s;\alpha_2,\mathbf{Q}^\sharp)$?

If there exists a solution to this problem denoted by $\widehat{\mathbf{Q}}$, then the quarter-car model in Fig. 1 with $c_t = 0.1c_s$ will have the closed-loop responses $\mathbf{H}_k(s;\alpha_2,\mathbf{Q}^\sharp)$, $k=1,2,3$ using the unique controller $\mathbf{K}$ corresponding to this $\widehat{\mathbf{Q}}$. Unfortunately, the formulated problem has no solution. To see this, first obtain the complete interpolation conditions for $\mathbf{T}_{\mathbf{z}_1V_i}$ from Eq. (46) as follows:

(i) $\mathbf{H}_1(s) = (c_sc_t/m_sm_u)s^{-1} + O(s^{-2})$,
(ii) $\mathbf{H}_1(0) = \mathbf{H}_1''(0) = 0,\ \mathbf{H}_1'(0) = 1$.

Then, from the requirement formulated above and (i):

$$s\,\mathbf{H}_1(s;\alpha_1,\widehat{\mathbf{Q}})|_{s=\infty} = s\,\mathbf{H}_1(s;\alpha_2\mathbf{Q}^\sharp)|_\infty,$$

which forces α_1 equal to α_2. Hence, there does not exist any solution.

Having seen the infeasibility of this interpolation problem, consider now the following variant:

Does there exist any $\widehat{\mathbf{Q}} \in \mathscr{R}H_\infty$ *such that* $\mathbf{H}_1(s;\alpha_1,\widehat{\mathbf{Q}}) = \mathbf{H}_1(s;\alpha_2,\mathbf{Q}^\sharp)\Psi(s)$ *for some* $\Psi \in \mathscr{R}H_\infty$?

Fortunately, there exists a solution to the latter problem. In fact, from the interpolation conditions for $\mathbf{T}_{\mathbf{z}_1V_i}$, it suffices to pick any $\Psi \in \mathscr{R}H_\infty$ satisfying

(iv) $\Psi(0) = 1$,
(v) $\Psi'(0) = 0$,
(vi) $\Psi(\infty) = \alpha_1/\alpha_2$.

It is easy to see that the following transfer function:

$$\Psi(s) = \frac{\sigma s^2 + as + b}{s^2 + as + b}, \qquad a,b>0,$$

where $\sigma = \alpha_1/\alpha_2$ has the aforementioned properties. Furthermore, for a given Ω which is sufficiently larger than the wheel-hop frequency ω_n^{wh}, if a and b are chosen so that $\Psi(\mathrm{j}\omega)$ is a good approximation to the low-pass filter:

$$L_\Omega(\omega) = \begin{cases} 1, & 0\leqslant\omega\leqslant\Omega, \\ \sigma, & \omega>\Omega \end{cases}$$

on the frequency band $[0,\Omega]$, then a good match to the vertical acceleration response plotted in Fig. 6 by the solid line is obtained. From the continuity of the trade-off curves, it follows that the other two responses are satisfactory as well.

It remains to calculate $\widehat{\mathbf{Q}}$. To this end, from Eq. (46),

$$\widehat{\mathbf{Q}} = \frac{\mathbf{H}_1(s;\alpha_2,\mathbf{Q}^\sharp)\Psi(s)\Delta(s;\alpha_1) - s(\alpha_1c_ss+k_t)(c_ss+k_s)}{m_ss^3(\alpha_1c_ss+k_t)(m_us^2+\alpha_1c_ss+k_t)}\ \Delta(s:\alpha_1), \tag{47}$$

where $\Delta(s:\alpha_1)$ is calculated from Eq. (8) with $c_t = \alpha_1c_s$. Since the degree of $\mathbf{H}_1(s;\alpha_2,\mathbf{Q}^\sharp)$ is 8, the degree of $\widehat{\mathbf{Q}}(s)$ is bounded above by 13. The numerator polynomial of $\widehat{\mathbf{Q}}$ before cancellations has order 18. Recall how $\Psi(\infty)$ was selected. This drops the order of the numerator polynomial by two. Three more degrees are canceled by the denumerator factor s^3. The end result is a proper transfer function $\widehat{\mathbf{Q}}$. Finally, $\mathbf{K}$ is calculated from Eq. (24) with $\widehat{\mathbf{Q}}$ in Eq. (47).

It may seem difficult to keep track of pole-zero cancellations. An easy way to circumvent this numerically ill-conditioned procedure is to evaluate $\widehat{\mathbf{Q}}(s)$ and/or its derivatives by computing the right-hand side of Eq. (47) and/or its-derivatives at a set of sufficiently many and arbitrarily selected frequencies s_k; and from these evaluations, obtain directly a minimal state-space realization of $\widehat{\mathbf{Q}}(s)$. For this purpose, numerically efficient robust algorithms developed in Refs. [33,34], which deal with multi-variable data as well, can be used.

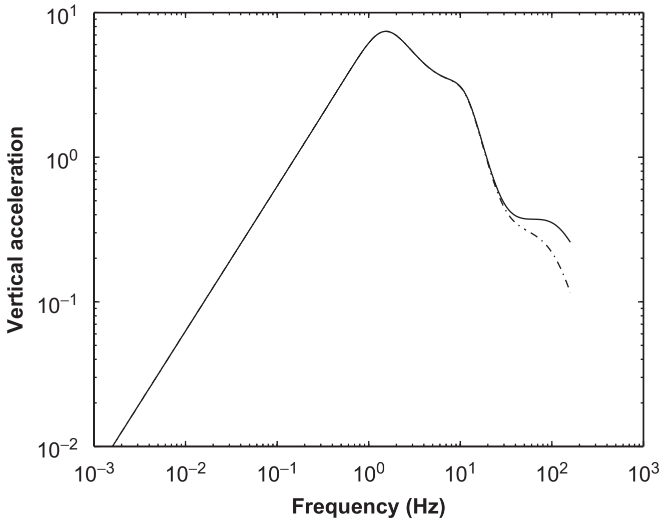

Fig. 10. The acceleration frequency response magnitude: (—) active suspension using the suspension travel measurement with the (fictitious) tire damping $c_t = 2c_s$; (–.) active suspension designed by a mixture of the LQG methodology and the interpolation approach using the suspension travel measurement with the (actual) tire damping $c_t = 0.1c_s$.

In Fig. 10, the acceleration frequency response magnitude of the active suspension designed by using the suspension travel measurement with the (fictitious) tire damping $c_t = 2c_s$ plotted in Fig. 6 is reproduced along with the acceleration frequency response magnitude of the active suspension designed by the hybrid algorithm outlined above. The purpose of plotting them together was to show that the approximation $\mathbf{H}_1(s;\alpha_1,\widehat{\mathbf{Q}}) \approx \mathbf{H}_1(s;\alpha_2,\mathbf{Q}^\sharp)$ is very accurate in the bandwidth of interest. This, in turn, implies that the approximations $\mathbf{H}_2(s;\alpha_1,\widehat{\mathbf{Q}}) \approx \mathbf{H}_2(s;\alpha_2,\mathbf{Q}^\sharp)$ and $\mathbf{H}_3(s;\alpha_1,\widehat{\mathbf{Q}}) \approx \mathbf{H}_3(s;\alpha_2,\mathbf{Q}^\sharp)$ are also very accurate in the same band of the frequencies. A comparison of Fig. 9 with Fig. 10 reveals impressive closed-loop performance enhancement by the interpolation approach. In the simulation, $\psi(s)$ was chosen as

$$\psi(s) = \frac{0.05s^2 + 500s + 10}{s^2 + 500s + 10}.$$

In Fig. 11, the actuator frequency response magnitudes of the active suspensions designed by the LOG methodology with tire dampings $c_t = 0.1c_s$ and $c_t = 2c_s$, and the hybrid algorithm with tire damping $c_t = 0.1\,c_s$ using the suspension travel measurement are plotted. Fig. 11 shows that the closed-loop performance enhancement by the interpolation approach is achieved at a reasonable price. Actually, the increase in the actuator gain is less than 30 decibels for all frequencies. Simulations for the values of tire damping at the equally spaced 101 points between and including $0.001\,c_s$ and $0.1\,c_s$ were also carried out. The numerical results plotted in Fig. 12 indicate that the actuator frequency response magnitudes of the active suspensions designed by either the LOG methodology or the hybrid algorithm using the suspension travel measurement are insentive to changes in tire damping; hence confirming the predication about the efficacy of coupling between the motions of the sprung and unsprung masses.

Moreover, in Fig. 11, the actuator gain is seen to peak at the heave and the wheel-hop frequencies, which indicates that a stable inversion of the vehicle transfer function is taking place by canceling these modes. This feature is reminiscent of the loop-transfer-recovery (LTR) synthesis [35] that applies to square and minimum-phase plants. Therefore, the hybrid algorithm can be viewed as a *loop-shaping* method realized in two-stages. The stages are the minimization of the quadratic criterion in Eq. (45) and the interpolation procedure. Although, in principle, it is possible to obtain a desired solution in one step by the LQG methodology, it is not clear how to accomplish this task since the quadratic criterion in Eq. (45) involves nine free weights in its most general form. Due to the coupling between the modes, it is a non-trivial matter to steer these weights towards a desired solution. The hybrid algorithm, on the other hand, ignores the interactions among the variables $\mathbf{z}_k,\ k = 1,2,3$ and u in the first stage. Then, the effect of the interactions is taken care of in the interpolation

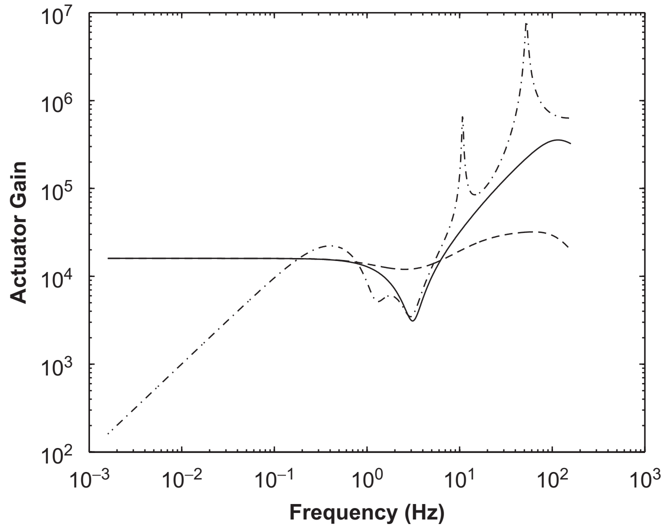

Fig. 11. The actuator frequency response magnitude using the suspension travel measurement: (—) the LQG design with tire damping $c_t = 2c_s$; (––) the LQG design with tire damping $c_t = 0.1c_s$; (–.) the hybrid algorithm with tire damping $c_t = 0.1c_s$.

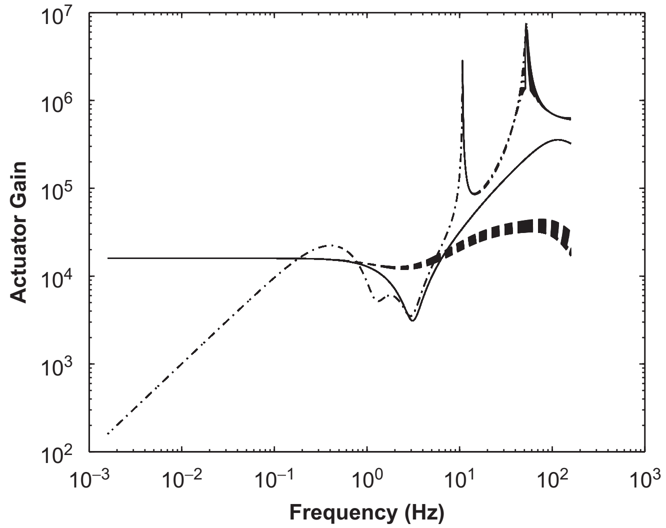

Fig. 12. The actuator frequency response magnitude using the suspension travel measurement: (—) the LQG design with tire damping $c_t = 2c_s$; (––) the LQG design with tire damping c_t at the equally spaced 101 points between and including $0.001c_s$ and $0.1c_s$; (–.) the hybrid algorithm with tire damping c_t at the same 101 points.

stage. The reader is cautioned not to draw broad conclusions based on this example solely since hardware limitations and uncertainties, in particular uncertainty in the tire model, may degrade actuator performance.

6. Conclusions

In this paper, the flexibility of shaping the closed-loop road frequency responses of a quarter-car model by feedback control was investigated. The constraints on the achievable responses of the quarter-car active suspension systems were derived for a wide range of the suspension parameters. The derived constraints complement the existing results in the literature on vehicle dynamics and control. Also, using the factorization approach of the feedback stability, it was shown that tire damping by coupling the motions of the sprung and unsprung masses eliminates a constraint on the wheel-hop mode. The influence of tire damping on the design of an active suspension for a lightly damped quarter-car model by a mixture of the LQG methodology and the interpolation approach was also illustrated.

The study of the constraints on the achievable performance has remained largely restricted to pointwise constraints on the frequency responses while ride comfort and safety criteria are mostly expressed in terms of the RMS values of the related transfer functions. Hence, a study of the constraints on the achievable RMS responses warrants future research.

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